Indian Statistical Institute, Bangalore
M. Math.
Second Year, Second Semester
Advanced Linear Algebra

Final examination | Date : April 25, 2025
Total Marks: 105 | Time: 3 hours
Maximum marks: 100 | Instructor: B V Rajarama Bhat

(1) Obtain polar and singular value decompositions for following matrices.

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \; Y = \begin{bmatrix} (-1) & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Demonstrate that polar decomposition need not be unique. [15]

(2) Let $\alpha : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a doubly stochastic completely positive map. Let A be a self-adjoint matrix in $M_n(\mathbb{C})$ and $B = \alpha(A)$. Show that $\lambda(B)$ is majorized by $\lambda(A)$, where for any matrix M, $\lambda(M)$ is the vector formed by eigenvalues of M in some order.

[15]

(3) Let $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ be a positive linear functional satisfying $\varphi(I) = 1$. Show that there exists a density matrix $\rho \in M_n(\mathbb{C})$ such that

$$\varphi(X) = \text{ trace } (\rho X), \; \forall X \in M_n(\mathbb{C}).$$

[15]

(4) Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

Find a unit vector z such that

$$\langle z, q(A)z \rangle = \frac{1}{3} \text{ trace}(q(A)),$$

for every polynomial q.

[15]

P.T.O.

(5) A graph is said to be a *tree* if it is connected and acyclic (without cycles). Suppose Q is the incidence matrix of a tree G with n-vertices where $n \geq 2$. Suppose $\hat{Q}$ is a matrix got by dropping a row of Q. Show that $\hat{Q}$ is non-singular. [15]

(6) Let G be a graph and let L be the Laplacian of G and let A be the adjacency matrix of G. Denote the largest eigenvalue of L by δ_1, the largest eigenvalue of A by λ_1 and let Δ be the maximal degree of G. Show (i) $\Delta \leq \delta_1$; (ii) $\lambda_1 \leq \Delta$. [15]

(7) Fix a natural number $n \geq 2$ and let $S = \{1, 2, \ldots, n\}$. Consider a graph $G = (V, E)$, where the vertex set V is the collection of all subsets of S. Two subsets A, B with $A \neq B$, form an edge iff A and B differ by atmost two elements, that is the number of elements in the symmetric difference $(A \bigcap B^c) \bigcup (A^c \bigcap B)$ is atmost two.

(i) What is the total number of edges in this graph? Is this graph bipartite?

(ii) Find the largest eigenvalue of the adjacency matrix of the graph.

(iii) For $n = 3$, write down the Laplacian matrix of the graph.

[15]

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